appear to be relatively independent of the nominal radius of the shell. Based on the local curvature of the "flat spot" rather than on the nominal curvature which is commonly used, a new upper bound may be determined for the collapse strength of an initially imperfect spherical shell. This observation is the basis of an analysis developed in Ref. 14 which adequately predicts both the elastic and inelastic strength of 36 machined models with local "flat spots" covering a relatively wide range of θ.

References

1 Feodosiev, V. I., "Calculation of thin clicking membranes," Prikl. Mat. Mekh., **X**, 295 (1946).

2 Budiansky, B., "Buckling of clamped shallow spherical shells," *Proceedings of the IUTAM Symposium of the Theory of Thin Elastic Shells* (North Holland Publishing Co., Amsterdam, 1960), p. 64.

3 Weinitschke, H., "On the stability problem for shallow spherical shells," J. Math. Phys. **38**, 209 (January 1960).

4 Thurston, G. A., "A numerical solution of the non-linear equations for axisymmetric bending of shallow spherical shells," J. Appl. Mech. **28**, 557 (December 1961).

5 Kloppel, K. and Jungbluth, O., "Beitrag zum Durchschlagsproblem dunwandieger Kugelschalen (Versuche und Bemessungsformeln)," Der Stahlbau, **22**, 121 (1953).

6 Tsien, H. S., "A theory for the buckling of thin shells," J. Aeronaut. Sci. **9**, 373 (August 1942).

7 Kaplan, A. and Fung, Y. C., "A non-linear theory of bending and buckling of thin elastic shallow spherical shells," NACA TN 3212 (August 1954).

8 Homewood, R. H., Brine, A. C., and Johnson, A. E., Jr., "Experimental investigation of the buckling instability of monocoque shells," Exptl. Mech. **1**, 88 (March 1961).

9 Chen, W. L., "Effect of geometrical imperfections on the elastic buckling of shallow spherical shells," Sc.D. Thesis, Mass. Inst. Tech. (January 1959).

10 Weinitschke, H., "Asymmetric buckling of clamped shallow spherical shells," NASA TN D-1510, 481 (December 1962).

11 Huang, N. C., "Unsymmetrical buckling of thin shallow spherical shells," Tech. Rept. 15, Harvard Univ. (March 1963).

12 Timoshenko, S., *Theory of Elastic Stability* (McGraw-Hill Book Co., Inc., New York, 1936), p. 491.

13 Krenzke, M. A., "The elastic buckling strength of near-perfect deep spherical shells with ideal boundaries," David Taylor Model Basin Rept. 1713 (July 1963).

14 Krenzke, M. A. and Kiernan, T. J., "The effect of initial imperfections on the collapse strength of deep spherical shells," David Taylor Model Basin Rept. 1757 (in preparation).

Some Observations on the Nonlinear Vibration of Thin Cylindrical Shells

David A. Evensen*

California Institute of Technology, Pasadena, Calif.

Recent works[1, 2] on the nonlinear vibration of cylindrical shells indicated that 1) the nonlinearity was of the "hardening" type, and 2) in some cases the problem was rather strongly nonlinear.

Suspecting that these phenomena would be readily detectable in the laboratory, the author performed a few experiments in which shells were vibrated at amplitudes of three to four wall thicknesses. The experimental results indicated that 1) the nonlinearity was of the "softening" type, and, 2) for the shells that were tested, the vibrations were only slightly nonlinear.

This led to a re-examination of the analysis. Using the functions for the displacement normal to the shell w and the

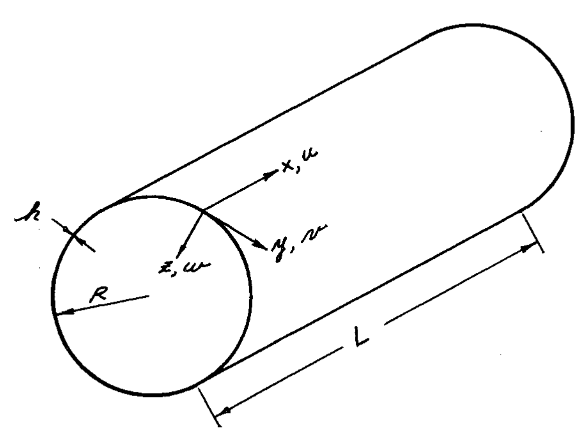

Fig. 1 Shell geometry and co-ordinate system.

stress function F from Ref. 1, it appears impossible to satisfy the constraint that the midplane circumferential displacement v be continuous and single-valued. That is, using a common notation (see Fig. 1), one cannot satisfy

$$\oint \frac{\partial v}{\partial y}\, dy = 0 \tag{1}$$

for all x with w and F as given by Chu.

Reissner, in his earlier work,[3] isolated a segment or "lobe" of a cylindrical shell and examined its nonlinear behavior. He implied that his results were applicable to complete cylindrical shells, but the continuity condition (1) was not satisfied. It is not surprising that his results are similar to those of Cummings[4] who studied curved panels. It may be noted in passing that nonlinear static stability analyses have made use of (1) since the 1941 paper of von Karman and Tsien.[5]

After some preliminary investigations, it was found that a deflection function of the form

$$w(x,y,t) = A(t)\sin\frac{m\pi x}{L}\sin\frac{ny}{R} + \frac{n^2A^2(t)}{4R}\sin^2\frac{m\pi x}{L} \tag{2}$$

together with the associated stress function, will satisfy the constraint (1) on v and still give zero normal displacement at the ends of the shell (at $x = 0$, $x = L$). With this combination of w and F, results were obtained that are in qualitative agreement with the experiments. The derivation was performed first using Galerkin's procedure with a weighting function given by $\partial w/\partial A$, and then the calculations were checked by employing the energy method. As in the previous papers, the author utilized the well-known shallow shell equations.

Shortly thereafter, the paper by Nowinski[2] appeared, confirming Chu's results and satisfying condition (1). It is of interest to note that Chu satisfies $w = 0$ at the ends of the shell but not the constraint (1) on v, whereas Nowinski satisfies the latter condition but not the former. Yet, they arrived at virtually identical results (for the isotropic case). This seems somewhat surprising at first, but possibly the answer lies in the fact that most approaches to date tacitly ignore the complementary solutions of the compatability equation

$$\frac{\nabla^4 F}{Eh} = -\frac{w_{xx}}{R} + [w_{xy}^2 - w_{xx}w_{yy}] \tag{3}$$

where

$$w_{xx} = \frac{\partial^2 w}{\partial x^2}, \text{etc.}$$

In fact, by adding a nonzero solution of $\nabla^4 F = 0$ to the stress function given by Chu, it is possible to satisfy (1) and still obtain solutions quite similar to his original results. It also

Received August 30, 1963.

* Graduate Student, Graduate Aeronautical Laboratories.

may be noted that if Nowinski had used an energy approach, his results might tend to agree with those of the writer.

Thus it appears that slightly different formulations of the "same" problem can lead to significantly different results. Additional experiments and refinements of the analysis are in progress at Graduate Aeronautical Laboratories, California Institute of Technology, and a full report will be issued on completion of the investigations. At this writing, however, it is safe to say that 1) the constraint on v should be satisfied in all problems involving the deflections of complete cylindrical shells, and 2) the complementary solutions of the compatibility equation (3) warrant further discussion.

References

[1] Chu, H. N., "Influence of large amplitudes on flexural vibrations of a thin circular cylindrical shell," J. Aerospace Sci. **28**, 602–609 (1961).

[2] Nowinski, J., "Nonlinear transverse vibrations of orthotropic cylindrical shells," AIAA J. **1**, 617–620 (1963).

[3] Reissner, E., "Nonlinear effects in vibrations of cylindrical shells," Ramo-Wooldridge Corp. Rept. AM5-6 (September 1955).

[4] Cummings, B. E., "Large amplitude vibration and response of curved panels," Inst. for Defense Analyses Res. Paper P-13 Washington, D. C. (March 1963).

[5] von Kármán, T. and Tsien, H. S., "The buckling of thin cylindrical shells under axial compression," J. Aeronaut. Sci. **8**, 303–312 (June 1941).

Optimum Launching to Hyperbolic Orbit by Two Impulses

STEPHEN LUBARD*

Polytechnic Institute of Brooklyn, Brooklyn, N. Y.

In this analysis the optimum launching of an object into a nonintersecting hyperbolic orbit is investigated. The effects of rotation and atmospheric drag were not included. The conclusions of this analysis can also be applied to the optimum landing on a planet from a hyperbolic orbit. Only the planar transfer is investigated, since this is always more economical than the nonplanar transfer.[1]

For the case of equal specific impulses, the problem of optimizing the launching is equivalent to minimizing the total velocity change I,

$$I = |\mathbf{V}_0| + |\mathbf{u}_2 - \mathbf{u}_1| \quad (1)$$

subject to the constraints of conservation of angular momentum and energy:

$$h_1 = V_0 \cos\omega_0 = u_1\rho \cos(\omega_2 + z) \quad (2)$$

$$h_2 = u_2\rho \cos\omega_2 \quad (3)$$

$$E_1 = \tfrac{1}{2}V_0^2 - 1 = \tfrac{1}{2}u_1^2 - 1/\rho \quad (4)$$

$$E_2 = \tfrac{1}{2}u_2^2 - 1/\rho \quad (5)$$

where subscripts 0, 1, and 2 refer to the launching, transfer orbit, and final hyperbola, as shown in Fig. 1. The radius of the planet and the gravitational acceleration have been equated to unity. With the four constraint equations we must optimize I with respect to three variables. Following Ref. 2, I will be optimized with respect to z, ω_0, and u_2.

Received August 28, 1963. The author wishes to acknowledge Lu Ting for his guidance in preparing this analysis.

* Student, participant in the National Science Foundation Science Education Program.

Substituting Eq. (4) into Eq. (1), one gets

$$I = \left[2 + u_1^2 - \frac{2}{\rho}\right]^{1/2} + [u_1^2 + u_2^2 - 2u_1u_2 (\cos z)]^{1/2} \quad (6)$$

which, for a given ρ and u_1, will be a minimum if $z = 0$. Therefore, the transfer trajectory should be tangent to the final hyperbola when firing the second impulse.

Equation (1) with $z = 0$ becomes

$$I = V_0 + |u_2 - u_1| \quad (7)$$

In order to remove the absolute value sign, we must show $u_2 \geq u_1$. Equations (3–5) yield

$$h_2^2 - 2E_2 - 2 = u_2^2(\rho^2 \cos^2\omega_2 - 1) - 2[1 - (1/\rho)] = [p_2 + (e_2 - 1)][p_2 - (e_2 + 1)]/p_2 \geq 0 \quad (8)$$

since the eccentricity e_2 and the perigee $p_2/(e_2 + 1)$ of the final hyperbola cannot be less than unity.

Similarly,

$$h_1 - 2E_1 - 2 = u_1^2(\rho^2 \cos^2\omega_2 - 1) - 2[1 - (1/\rho)] = [p_1 + (e_1 - 1)][p_1 - (e_1 + 1)]/p_1 \leq 0 \quad (9)$$

since the perigee $p_1/(e_1 + 1)$ of the transfer orbit cannot be greater than unity.

It follows that $u_2 \geq u_1$. Substituting Eq. (2) with $z = 0$ into Eq. (7) yields

$$I = \frac{u_1\rho \cos\omega_2}{\cos\omega_0} + (u_2 - u_1) \quad (10)$$

For a given u_2 and ρ we may optimize I with respect to ω_0 by taking the first derivative,

$$\frac{dI}{d\omega_0} = \left[\frac{u_1\rho \cos\omega_2}{\cos\omega_0(\rho \cos\omega_2 + \cos\omega_0)}\right] \sin\omega_0 \quad (11)$$

Equation (11) equals zero if, and only if, $\omega_0 = 0$ for $0 \leq \omega_0 \leq \pi/2$. Since $(d^2I/d\omega_0^2) > 0$ at $\omega_0 = 0$, I is a minimum for a tangential launching ($\omega_0 = 0$).

I must now be minimized with respect to u_2. Substituting from Eqs. (2–5), with $z = \omega_0 = 0$ into Eq. (10), one obtains

$$I = h_2 \left[\frac{2 + 2E_2 - u_2^2}{h_2^2 - u_2^2}\right]^{1/2} + u_2 \left[1 - \left(\frac{2 + 2E_2 - u_2^2}{h_2^2 - u_2^2}\right)^{1/2}\right] \quad (12)$$

Differentiating (12), one gets

$$\frac{dI}{du_2} = F(h_2 - u_2)[h_2(2 + 2E_2 - u_2^2)^{1/2} - u_2(h_2^2 - u_2^2)^{1/2}]$$

where

$$F = \frac{(h_2^2 - u_2^2)^{1/2} - (2 + 2E_2 - u_2^2)^{1/2}}{(h_2^2 - u_2^2)^{3/2}(2 + 2E_2 - u_2^2)^{1/2}}$$

From Eq. (8) we know that $F > 0$. Therefore, $dI/du_2 = 0$ if,

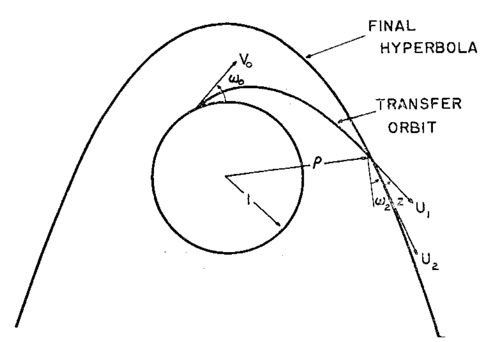

Fig. 1 Launching to hyperbolic orbit.